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MINIMUM PAIR-DEGREE FOR TIGHT HAMILTONIAN CYCLES IN 4-UNIFORM HYPERGRAPHS

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Abstract. We show that every 4-uniform hypergraph with n vertices and minimum pair-degree at least $(5/9+o(1))n^2/2$ contains a tight Hamiltonian cycle. This degree condition is asymptotically optimal. In the proof we use a variant of the absorbing method and ideas from the proof of the optimal minimum vertex degree condition for tight Hamiltonian cycles in 3-uniform hypergraphs that was obtained in a previous work by Reiher, Rödl, Ruciński, Schacht, and Szemerédi.

1. Background and main result

We deal with hypergraph extensions of Dirac's Theorem. In 1952 G. A. Dirac [**1**] proved that every graph $G = (V, E)$ on at least 3 vertices and with minimum vertex degree $\delta(G) \geq |V|/2$ contains a Hamiltonian cycle. This result is best possible, as there are graphs G with minimum degree $\delta(G) = \lceil |V|/2 \rceil - 1$ not containing a Hamiltonian cycle.

For $k \geq 2$, a *k-uniform hypergraph*, or, shortly, a *k-graph*, is a pair (V, E), where $E \subseteq V^{(k)} := \{e \subseteq V : |e| = k\}$, that is, the edge set $E = E(H)$ consists of k-element sets of vertices. After several earlier, related results offering various Dirac-type conditions ensuring the existence of Hamiltonian cycles in k-graphs (see, e.g., [**6**] and [**9**]), in [**5**] the following extension of Dirac's Theorem was established.

A 3-graph H is called *Hamiltonian* if for some cyclic ordering of its vertices $V(H) = \{x_1, \dots, x_n\}$, every consecutive triple of vertices $\{x_i, x_{i+1}, x_{i+2}\}$ with $i \in \mathbb{Z}/n\mathbb{Z}$ is an edge of H.

Theorem 1.1. *For every $\alpha > 0$ there exists an integer n_0 such that every 3-uniform hypergraph H with $n \geq n_0$ vertices and with minimum vertex degree $\delta(H) \geq \left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$ is Hamiltonian.*

One may consider various further extensions of Theorem 1.1 to k-uniform hypergraphs. Perhaps the most natural one would be to find the smallest constant c_k with the property that, for large n, every n-vertex k-graph H with

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minimum degree $\delta(H) \geq (c_k + o(1))n^{k-1}/(k-1)!$ is Hamiltonian. (Similarly, as for $k = 3$, a k-graph H is called *Hamiltonian* if for some cyclic ordering of its vertices $V(H) = \{x_1, \dots, x_n\}$, every consecutive k-element segment of vertices $\{x_i, x_{i+1}, \dots, x_{i+k-1}\}$ with $i \in \mathbb{Z}/n\mathbb{Z}$ is an edge of H.)

Since finding an optimal value of the parameter c_k seems beyond the reach of our current methodology, here we take another approach, which allows us to utilise some of the methods developed in [**4**] and [**5**]. Given a k-graph $H = (V, E)$ and a subset $S \subseteq V$, we denote by $d_H(S)$ *the degree of* S *in* H, that is, the number of edges $e \in H$ with $S \subseteq e$. For $d \in [k-1]$, *the minimum* d*-degree* $\delta_d(H)$ is the smallest value of $d_H(S)$ taken over all d-element subsets $S \subseteq V$. Finally, given d, k, and n, with $1 \leq d \leq k-1 < n$, we define

$$h_d^{(k)}(n) = \min\{h \in \mathbb{N} : \text{each } n\text{-vertex } k\text{-graph } H \text{ with } \delta_d(H) \geq h \text{ is Hamiltonian}\}.$$

The following generalisation of Dirac's result was proved in [**7**].

Theorem 1.2. *For every $k \geq 2$, we have $h_{k-1}^{(k)}(n) = \left(\frac{1}{2} + o(1)\right) n$.*

Here we concentrate on the next value of d, namely $d = k-2$, and formulate the following conjecture.

Conjecture 1.3. For all $k \geq 3$, we have $h_{k-2}^{(k)}(n) = \left(\frac{5}{9} + o(1)\right) n^2/2$.

The construction presented later in this section shows that, if true, this conjecture provides the best possible constant. In [**5**] Conjecture 1.3 was verified for $k = 3$. The main result of this paper establishes it for $k = 4$. For $k \geq 5$ it remains open.

Theorem 1.4 (Main Theorem)**.** *For every $\alpha > 0$ there exists an integer n_0 such that every 4-uniform hypergraph H with $n \geq n_0$ vertices and with minimum pair-degree $\delta_2(H) \geq \left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$ is Hamiltonian.*

2. Tight paths and cycles

For $k \geq 3$, a k-graph P is a *tight path of length* ℓ, if $|V(P)| = \ell + k - 1$ and there is an ordering of the vertices $V(P) = \{x_1, \dots, x_{\ell+k-1}\}$ such that a k-element subset e forms an edge of P if and only if $e = \{x_i, x_{i+1}, \dots, x_{i+k-1}\}$ for some $i \in [\ell]$. The ordered $(k-1)$-tuples $(x_1, x_2, \dots, x_{k-1})$ and $(x_{\ell+1}, x_{\ell+2}, \dots, x_{\ell+k-1})$ are the *end-$(k-1)$-tuples* of P and we say that P is a tight path from $(x_1, x_2, \dots, x_{k-1})$ to $(x_{\ell+1}, x_{\ell+2}, \dots, x_{\ell+k-1})$. This definition of end-tuples is not symmetric and implicitly fixes a direction on P and the order of the end-tuples. For $k = 3$ we call the end-tuples *end-pairs*, and for $k = 4$ *end-triples.* All other vertices of P are called *internal.* We sometimes identify such a path P with the sequence of its vertices $x_1 \dots x_{\ell+k-1}$.

Furthermore, a *tight cycle C of length $\ell \geq k+1$* consists of a path $x_1 \dots x_\ell$ of length $\ell-k+1$ and $k-1$ additional edges $\{x_{\ell-k+1}, \dots, x_\ell, x_1\}$, $\dots$, $\{x_\ell, \dots, x_{k-1}\}$. In both cases the *length* of a tight cycle and of a tight path is measured by the number of edges. For simplicity, when $k = 3$ or $k = 4$, we denote edges by xyz and $xyzw$ instead of $\{x, y, z\}$ and $\{x, y, z, w\}$.

If a tight cycle C is a sub-k-graph of another k-graph H with the same number of vertices, then we call C a *tight Hamiltonian cycle in H*. Note that with our earlier notion H is *Hamiltonian* if and only if H contains a tight Hamiltonian cycle. Throughout this abstract, we will skip the word 'tight', as we are not considering any other types of cycles.

3. Lower bound

For $k \geq 4$, we provide a construction which shows that the degree constraint in Conjecture 1.3 is asymptotically optimal. Our construction is based on a bipartition of the vertex set and forbidding just one type of edges to be present, similarly as it was done for $k = 3$ in [**5**] (see also [**3**] and [**7**]). For $k \geq 4$ and sufficiently large n, let $|V| = n$, $V = X\dot{\cup}Y$, and $|X| = \lfloor \frac{2}{3}n \rfloor$. Further, let j be an integer such that

$$\tfrac{2}{3}k - 1 < j < \tfrac{2}{3}k + 1.$$

Define a k-graph H on V with edge set $V^{(k)} \setminus E_j$, where E_j is the set of all k-element subsets of V with exactly j vertices in X (and thus exactly $k-j$ vertices in Y). We need to show that

(a) there is no Hamiltonian cycle in H, and

(b) $\delta_{k-2}(H) \sim \frac{5}{9}\frac{n^2}{2}$.

To prove (a), suppose to the contrary that there is a Hamiltonian cycle C in H and consider the quantity $Q = \sum_{e \in C} |e \cap X|$. As every vertex belongs to precisely k edges of C, we have $Q = k|X| = k\lfloor \frac{2}{3}n \rfloor$. Thus, by averaging, there exist edges $e_1, e_2 \in C$ such that $|e_1 \cap X| \leq \frac{2}{3}k < j + 1$ and $|e_2 \cap X| \geq \frac{2}{3}k > j - 1$. But, by our construction, there are no edges $e \in H$ with $|e \cap X| = j$. Thus, in fact, $|e_1 \cap X| \leq j - 1$ and $|e_2 \cap X| \geq j + 1$. However, this is obviously impossible in view of the lack of edges in C with precisely j vertices in X. We omit the calculations that prove (b).

4. Overview of the proof

The proof relies on the absorption method introduced in [**8**]. We will construct a large cycle covering almost all vertices with the property that it can absorb the remaining vertices into it, i.e. that we can build a cycle containing both the vertices of the previous cycle and the remaining vertices. Firstly, we show that for every quadruple of vertices $v_1v_2v_3v_4$ there are many subgraphs with a certain structure; we will call these subgraphs $v_1v_2v_3v_4$-absorbers. A $v_1v_2v_3v_4$-absorber consists of vertices which can build paths together with $v_1v_2v_3v_4$ and without $v_1v_2v_3v_4$ such that these paths have the same end-triples and contain all vertices of the absorber. With the probabilistic method we can then find a small set of absorbers such that every quadruple $v_1v_2v_3v_4$ has many $v_1v_2v_3v_4$-absorbers inside this set. Further, we will show that we can in fact connect the absorbers in this set to an absorbing path P_A containing only few vertices. Due to the structure of absorbers, this path has the property that for any set X of not too many vertices, there is a path on the vertex set $V(P_A) \cup X$ and with the same end-triples as P_A. Since $|P_A|$ is small, the degree condition stays almost intact in $H - P_A$. Next, we show that we can cover almost all vertices with one long path and connect the end-triples of this path to the end-triples of P_A creating a cycle. Lastly, we can absorb the few remaining vertices into P_A, leaving the end-triples and therefore the connections to the long path intact, yielding a Hamiltonian cycle.

Let us look at some of these steps in a little more detail. For some of them, the idea is to look into the link graphs of vertices, and make use of the minimum vertex degree condition with methods from [**5**]. (The link graph of a vertex u in a 4-graph H is a 3-graph on $V(H)$ with xyz being an edge if and only if $uxyz$ is an edge in H.) Firstly, at several points we needed to connect two end-triples of paths by a path. We can only guarantee such a connection for certain, so called *connectable*, triples. Roughly speaking, a triple abc is connectable if for many vertices u, the pairs ab and bc can be connected by many tight 3-uniform paths in the link graph of u. To prove such a Connecting Lemma, i.e., to construct many paths connecting two triples abc and xyz, the basic idea is as follows. We show that there is a large set U of vertices in whose link graphs we find a common large set of paths connecting bc with xy. Each of those 3-uniform paths gives rise to many 4-uniform paths when we insert vertices from U at every fourth position. So in some sense 4-graphs with the minimum pair-degree condition from Theorem 1.4 “inherit” their connectivity from the connectivity of 3-graphs with a respective minimum vertex degree condition. In fact, this is an example of a more general strategy of proving a Connecting Lemma by using good connectivity properties in the link graphs. A simple probabilistic argument now ensures the existence of a small reservoir set such that all connectable triples are in fact connectable by many paths taking all their internal vertices from this reservoir.

As mentioned above, following an idea of [**4**] we will always absorb four vertices into one absorber, whereas commonly a single vertex is absorbed into one absorber. The advantage of our approach is that we can then use absorbers whose main part is a 4-partite 4-graph. When showing that for each quadruple there exist many absorbers, we can then make use of a result by Erdös [**2**] that 4-partite 4-graphs have Turán density 0 and that hence, by supersaturation, there exist many copies of any small 4-partite 4-graph in a 4-graph satisfying the minimum pair-degree condition in Theorem 1.4. The divisibility issues arising from this way of absorbing are fairly easy to deal with.

Lastly, let us sketch the construction of the almost covering path. While proofs for the existence of almost covering subgraphs often rely on the Hypergraph Regularity Method, we are indeed able to finish without using it. We will instead argue that a maximal path Q consisting of a set $\mathcal{P}$ of paths (with certain properties) that are connected through the reservoir will cover almost everything. To do this, we will assume that the set of uncovered vertices U is large and construct a longer path. For that we will not only use the uncovered vertices but also vertices of some of the paths in $\mathcal{P}$. Since we will be able to construct more new paths to add to $\mathcal{P}$ than we have to take out, in the end we get a longer path. By the probabilistic method we will show that there is some selection $\mathcal{P}' \subseteq \mathcal{P}$ such that for many vertices from U the link graph induced on $V(\mathcal{P}')$ satisfies the minimum vertex degree condition for Hamiltonian cycles in 3-graphs (note that the proof in [**5**] does not use the Regularity Lemma). In fact, we only need some covering of almost all vertices in this link graph with certain (3-uniform) paths. By inserting some vertices from U at every fourth position into these paths we get more than $|\mathcal{P}'|$ new 4-uniform paths. Then we get a longer path from Q by taking out the paths in $\mathcal{P}'$, adding the newly constructed paths and connecting everything through the reservoir.

The fact that important parts of the proof can in parts be reduced to the result on the minimum vertex degree condition for 3-uniform hypergraphs together with the lower bound constructions motivate Conjecture 1.3.

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